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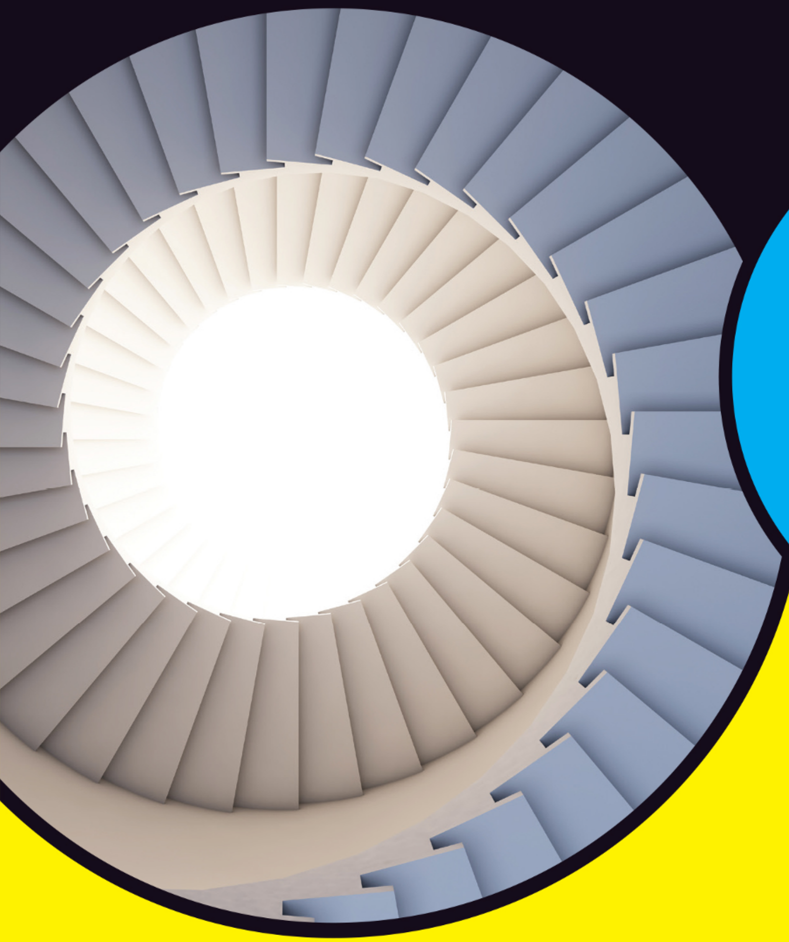

In-depth review of key concepts

Example problems for every lesson

Step-by-step explanations in plain English

Mary Jane Sterling

Author of *Algebra I For Dummies*

Algebra II

2nd Edition

by Mary Jane Sterling

for dummies
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Algebra II For Dummies®, 2nd Edition

Published by: **John Wiley & Sons, Inc.,** 111 River Street, Hoboken, NJ 07030-5774, www.wiley.com

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Library of Congress Control Number: 2018956041

ISBN 978-1-119-54314-5 (pbk); ISBN 978-1-119-54317-6 (ebk); ISBN 978-1-119-54310-7 (ebk)

Manufactured in the United States of America

10 9 8 7 6 5 4 3 2 1

Contents at a Glance

Introduction 1

Part 1: Homing in on Basic Solutions 5

CHAPTER 1: Going Beyond Beginning Algebra 7
CHAPTER 2: Toeing the Straight Line: Linear Equations 21
CHAPTER 3: Conquering Quadratic Equations 35
CHAPTER 4: Rooting Out the Rational, Radical, and Negative 55
CHAPTER 5: Graphing Your Way to the Good Life 73

Part 2: Facing Off with Functions 95

CHAPTER 6: Formulating Function Facts 97
CHAPTER 7: Sketching and Interpreting Quadratic Functions 115
CHAPTER 8: Staying Ahead of the Curves: Polynomials 133
CHAPTER 9: Reasoning with Rational Functions 157
CHAPTER 10: Exposing Exponential and Logarithmic Functions 177

Part 3: Conquering Conics and Systems of Equations 203

CHAPTER 11: Cutting Up Conic Sections 205
CHAPTER 12: Solving Systems of Linear Equations 229
CHAPTER 13: Solving Systems of Nonlinear Equations and Inequalities 251

Part 4: Shifting into High Gear with Advanced Concepts 271

CHAPTER 14: Simplifying Complex Numbers in a Complex World 273
CHAPTER 15: Making Moves with Matrices 287
CHAPTER 16: Making a List: Sequences and Series 307
CHAPTER 17: Everything You Wanted to Know about Sets 329

Part 5: The Part of Tens 353

CHAPTER 18: Ten Multiplication Tricks 355
CHAPTER 19: Ten Special Types of Numbers 365

Index 371

Table of Contents

INTRODUCTION 1
About This Book. 1
Foolish Assumptions 2
Icons Used in This Book 3
Beyond the Book 4
Where to Go from Here 4

PART 1: HOMING IN ON BASIC SOLUTIONS 5

CHAPTER 1: **Going Beyond Beginning Algebra** 7
Outlining Algebraic Properties. 8
Keeping order with the commutative property 8
Maintaining group harmony with the associative property 9
Distributing a wealth of values 9
Checking out an algebraic ID 10
Singing along in-verses 11
Ordering Your Operations 11
Zeroing in on the Multiplication Property of Zero 12
Expounding on Exponential Rules 13
Multiplying and dividing exponents 13
Getting to the roots of exponents. 14
Raising or lowering the roof with exponents 14
Making nice with negative exponents 15
Implementing Factoring Techniques 15
Factoring two terms 16
Taking on three terms. 17
Factoring four or more terms by grouping 19

CHAPTER 2: **Toeing the Straight Line: Linear Equations** 21
Linear Equations: Handling the First Degree. 21
Tackling basic linear equations 22
Clearing out fractions 23
Isolating different unknowns. 24
Linear Inequalities: Algebraic Relationship Therapy 25
Solving linear inequalities. 26
Introducing interval notation. 27
Compounding inequality issues 28
Absolute Value: Keeping Everything in Line 30
Solving absolute value equations 31
Seeing through absolute value inequality 31

CHAPTER 3: **Conquering Quadratic Equations** 35
Implementing the Square Root Rule 36
Dismantling Quadratic Equations into Factors 37
Factoring binomials 37
Factoring trinomials 39
Factoring by grouping 40
Resorting to the Quadratic Formula 41
Finding rational solutions 42
Straightening out irrational solutions 42
Formulating huge quadratic results 43
Completing the Square: Warming Up for Conics 43
Squaring up a quadratic equation 44
Completing the square twice over 45
Tackling Higher-Powered Polynomials 46
Handling the sum or difference of cubes 47
Tackling quadratic-like trinomials 48
Solving Quadratic Inequalities 49
Keeping inequality strictly quadratic 50
Signing up for fractions 52
Increasing the number of factors 53
Considering absolute value inequalities 53

CHAPTER 4: **Rooting Out the Rational, Radical, and Negative** 55
Acting Rationally with Fraction-Filled Equations 56
Systematically solving rational equations 56
Solving rational equations with proportions 60
Ridding Yourself of a Radical 61
Squaring both sides of a radical equation 62
Calming two radicals 63
Changing Negative Attitudes about Exponents 65
Flipping negative exponents out of the picture 65
Factoring out negatives to solve equations 66
Fooling Around with Fractional Exponents 68
Combining terms with fractional exponents 69
Factoring fractional exponents 69
Solving equations by working with fractional exponents 70

CHAPTER 5: **Graphing Your Way to the Good Life** 73
Coordinating Your Graphing Efforts 74
Identifying the parts of the coordinate plane 74
Plotting from dot to dot 75
Streamlining the Graphing Process with Intercepts and Symmetry 76
Finding x- and y-intercepts 77
Reflecting on a graph's symmetry 78

Graphing Lines . . . 80
Finding the slope of a line . . . 81
Facing two types of equations for lines . . . 82
Identifying parallel and perpendicular lines . . . 85
Looking at 10 Basic Forms . . . 86
Lines and quadratics . . . 86
Cubics and quartics . . . 87
Radicals and rationals . . . 87
Exponential and logarithmic curves . . . 88
Absolute values and circles . . . 89
Solving Problems with a Graphing Calculator . . . 89
Entering equations into graphing calculators correctly . . . 90
Looking through the graphing window . . . 92

PART 2: FACING OFF WITH FUNCTIONS . . . 95

CHAPTER 6: **Formulating Function Facts** . . . 97
Defining Functions . . . 98
Introducing function notation . . . 98
Evaluating functions . . . 98
Homing In on Domain and Range . . . 99
Determining a function’s domain . . . 99
Describing a function’s range . . . 100
Betting on Even or Odd Functions . . . 102
Recognizing even and odd functions . . . 102
Applying even and odd functions to graphs . . . 103
Facing One-to-One Confrontations . . . 104
Defining one-to-one functions . . . 104
Eliminating one-to-one violators . . . 105
Going to Pieces with Piecewise Functions . . . 106
Doing piecework . . . 107
Applying piecewise functions . . . 108
Composing Yourself and Functions . . . 110
Performing compositions . . . 110
Simplifying the difference quotient . . . 111
Singing Along with Inverse Functions . . . 112
Determining if functions are inverses . . . 112
Solving for the inverse of a function . . . 113

CHAPTER 7: **Sketching and Interpreting Quadratic Functions** . . . 115
Interpreting the Standard Form of Quadratics . . . 116
Starting with “a” in the standard form . . . 116
Following up with “b” and “c” . . . 117

Investigating Intercepts in Quadratics 118
Finding the one and only y-intercept 119
Finding the x-intercepts 120
Going to the Extreme: Finding the Vertex 123
Lining Up along the Axis of Symmetry 124
Sketching a Graph from the Available Information 125
Applying Quadratics to the Real World 127
Selling candles 127
Shooting basketballs 128
Launching a water balloon 130

CHAPTER 8: **Staying Ahead of the Curves: Polynomials** 133
Taking a Look at the Standard Polynomial Form 134
Exploring Polynomial Intercepts and Turning Points 134
Interpreting relative value and absolute value 135
Counting intercepts and turning points 135
Solving for polynomial intercepts 138
Determining Positive and Negative Intervals .139
Using a sign-line 140
Interpreting the rule 141
Finding the Roots of a Polynomial 143
Factoring for polynomial roots 143
Saving your sanity: The Rational Root Theorem 145
Letting Descartes make a ruling on signs .148
Synthesizing Root Findings 150
Using synthetic division to test for roots 150
Synthetically dividing by a binomial 153
Wringing out the Remainder (Theorem) 154

CHAPTER 9: **Reasoning with Rational Functions** 157
Exploring Rational Functions 158
Sizing up domain 158
Introducing intercepts 159
Adding Asymptotes to the Rational Pot 160
Determining the equations of vertical asymptotes 160
Determining the equations of horizontal asymptotes 161
Graphing vertical and horizontal asymptotes 161
Crunching the numbers and graphing oblique asymptotes 163
Accounting for Removable Discontinuities 164
Removal by factoring 164
Evaluating the removal restrictions 165
Showing removable discontinuities on a graph 165

Pushing the Limits of Rational Functions 167
Evaluating limits at discontinuities 168
Going to infinity 170
Catching rational limits at infinity 172
Putting It All Together: Sketching Rational Graphs from Clues 173

CHAPTER 10: **Exposing Exponential and Logarithmic Functions** 177
Evaluating Exponential Expressions 178
Exponential Functions: It's All about the Base, Baby 179
Observing the trends in bases 179
Meeting the most frequently used bases: 10 and e 180
Solving Exponential Equations 182
Making bases match 182
Recognizing and using quadratic patterns .184
Showing an "Interest" in Exponential Functions 186
Applying the compound interest formula 186
Looking at continuous compounding 188
Logging On to Logarithmic Functions 189
Meeting the properties of logarithms 190
Putting your logs to work 191
Solving Logarithmic Equations 193
Setting log equal to log 194
Rewriting log equations as exponentials 195
Graphing Exponential and Logarithmic Functions 196
Expounding on the exponential 196
Not seeing the logs for the trees 198

PART 3: CONQUERING CONICS AND SYSTEMS OF EQUATIONS 203

CHAPTER 11: **Cutting Up Conic Sections** 205
Cutting Up a Cone 206
Opening Every Which Way with Parabolas 206
Looking at parabolas with vertices at the origin 207
Observing the general form of parabola equations 210
Sketching the graphs of parabolas 211
Converting parabolic equations to the standard form 214
Going Round and Round in Conic Circles 215
Standardizing the circle 215
Specializing in circles 217
Preparing Your Eyes for Solar Ellipses 218
Raising the standards of an ellipse 218
Sketching an elliptical path 221

Feeling Hyper about Hyperbolas. 222
Including the asymptotes 223
Graphing hyperbolas 224
Identifying Conics from Their Equations, Standard or Not 227

CHAPTER 12: **Solving Systems of Linear Equations** 229
Looking at the Standard Linear-Systems Form and Its Possible Solutions. 230
Graphing Solutions of Linear Systems 230
Pinpointing the intersection. 231
Toeing the same line twice. 232
Dealing with parallel lines 232
Solving Systems of Two Linear Equations by Using Elimination. 233
Getting to the point with elimination 234
Recognizing solutions indicating parallel or coexisting lines 235
Making Substitution the Choice 236
Variable substituting made easy 236
Identifying parallel and coexisting lines 237
Using Cramer’s Rule to Defeat Unwieldy Fractions 238
Setting up the linear system for Cramer 239
Applying Cramer’s Rule to a linear system 240
Tackling Linear Systems with Three Linear Equations 241
Solving three-equation systems with algebra 241
Generalizing multiple solutions for linear equations 243
Upping the Ante with Larger Systems 244
Applying Linear Systems to Our 3-D World 247
Using Systems to Decompose Fractions 248

CHAPTER 13: **Solving Systems of Nonlinear Equations and Inequalities** 251
Crossing Parabolas with Lines 252
Determining the point(s) where a line and parabola cross paths 253
Dealing with a solution that’s no solution 254
Intertwining Parabolas and Circles 255
Managing multiple intersections. 256
Sorting out the solutions 258
Planning Your Attack on Other Systems of Equations 260
Mixing polynomials and lines 260
Crossing polynomials 261
Navigating exponential intersections 263
Rounding up rational functions. 265
Playing Fair with Inequalities 268
Drawing and quartering inequalities 268
Graphing areas with curves and lines 269

PART 4: SHIFTING INTO HIGH GEAR WITH ADVANCED CONCEPTS 271

CHAPTER 14: Simplifying Complex Numbers in a Complex World 273
Using Your Imagination to Simplify Powers of *i* 274
Understanding the Complexity of Complex Numbers 275
Operating on complex numbers 276
Multiplying by the conjugate to perform division 277
Simplifying radicals 279
Solving Quadratic Equations with Complex Solutions 280
Working Polynomials with Complex Solutions 282
Identifying conjugate pairs 283
Interpreting complex zeros 283

CHAPTER 15: Making Moves with Matrices 287
Describing the Different Types of Matrices 288
Row and column matrices 289
Square matrices 289
Zero matrices 289
Identity matrices 289
Performing Operations on Matrices 290
Adding and subtracting matrices 290
Multiplying matrices by scalars 291
Multiplying two matrices 291
Applying matrices and operations 293
Defining Row Operations 297
Finding Inverse Matrices 298
Determining additive inverses 299
Determining multiplicative inverses 299
Dividing Matrices by Using Inverses 304
Using Matrices to Find Solutions for Systems of Equations 305

CHAPTER 16: Making a List: Sequences and Series 307
Understanding Sequence Terminology 308
Using sequence notation 308
No-fear factorials in sequences 309
Alternating sequential patterns 309
Looking for sequential patterns 310
Taking Note of Arithmetic and Geometric Sequences 313
Finding common ground: Arithmetic sequences 313
Taking the multiplicative approach: Geometric sequences 315
Recursively Defining Functions 317

Making a Series of Moves . . . 318
Introducing summation notation . . . 318
Summing arithmetically . . . 319
Summing geometrically . . . 320
Applying Sums of Sequences to the Real World . . . 323
Stacking the blocks . . . 323
Negotiating your allowance . . . 323
Bouncing a ball . . . 324
Highlighting Special Formulas . . . 326

CHAPTER 17: **Everything You Wanted to Know about Sets** . . . 329
Revealing Set Notation . . . 329
Listing elements with a roster . . . 330
Building sets from scratch . . . 330
Going for all (universal set) or nothing (empty set) . . . 331
Subbing in with subsets . . . 331
Operating on Sets . . . 333
Celebrating the union of two sets . . . 333
Looking both ways for set intersections . . . 334
Feeling complementary about sets . . . 335
Counting the elements in sets . . . 335
Drawing Venn You Feel Like It . . . 336
Applying the Venn diagram . . . 337
Using Venn diagrams with set operations . . . 338
Adding a set to a Venn diagram . . . 339
Focusing on Factorials . . . 342
Making factorial manageable . . . 342
Simplifying factorials . . . 343
How Do I Love Thee? Let Me Count Up the Ways . . . 344
Applying the multiplication principle to sets . . . 344
Arranging permutations of sets . . . 345
Mixing up sets with combinations . . . 348
Branching Out with Tree Diagrams . . . 350
Picturing a tree diagram for a permutation . . . 351
Drawing a tree diagram for a combination . . . 352

PART 5: THE PART OF TENS . . . 353

CHAPTER 18: **Ten Multiplication Tricks** . . . 355
Squaring Numbers That End in 5 . . . 355
Finding the Next Perfect Square . . . 356
Recognizing the Pattern in Multiples of 9 and 11 . . . 357
Casting Out 9s . . . 357
Casting Out 9s: The Multiplication Moves . . . 358
Multiplying by 11 . . . 359

Multiplying by 5 .360
Finding Common Denominators. .361
Determining Divisors .362
Multiplying Two-Digit Numbers. .362

CHAPTER 19: **Ten Special Types of Numbers** .365
Triangular Numbers .365
Square Numbers .366
Hexagonal Numbers .366
Perfect Numbers .367
Amicable Numbers .367
Happy Numbers .368
Abundant Numbers .368
Deficient Numbers .368
Narcissistic Numbers .368
Prime Numbers .369

INDEX .371

Introduction

Here you are, contemplating reading a book on Algebra II. It isn't a mystery novel, although you can find people who think mathematics in general is a mystery. It isn't a historical account, even though you find some historical tidbits scattered here and there. Science fiction it isn't; mathematics is a science, but you find more fact than fiction. As Joe Friday (star of the old *Dragnet* series) says, "The facts, ma'am, just the facts." This book isn't light reading, although I attempt to interject humor whenever possible. What you find in this book is a glimpse into the way I teach: uncovering mysteries, working in historical perspectives, providing information, and introducing the topic of Algebra II with good-natured humor. This book has the best of all literary types! Over the years, I've tried many approaches to teaching algebra, and I hope that with this book I'm helping you cope with and incorporate other teaching methods.

About This Book

Because you're interested in this book, you probably fall into one of four categories:

- You're fresh off Algebra I and feel eager to start on this new venture.
- You've been away from algebra for a while, but math has always been your main interest, so you don't want to start too far back.
- You're a parent of a student embarking on or having some trouble with an Algebra II class and you want to help.
- You're just naturally curious about science and mathematics and you want to get to the good stuff that's in Algebra II.

Whichever category you represent (and I may have missed one or two), you'll find what you need in this book. You can find some advanced algebraic topics, but I also cover the necessary basics, too. You can also find plenty of connections — the ways different algebraic topics connect with each other and the ways the algebra connects with other areas of mathematics.

After all, the many other math areas drive Algebra II. Algebra is the passport to studying calculus, trigonometry, number theory, geometry, all sorts of good mathematics, and much of science. Algebra is basic, and the algebra you find here will help you grow your skills and knowledge so you can do well in math courses and possibly pursue other math topics.

To help you navigate this book, I use the following conventions:

- I italicize special mathematical terms and define them right then and there so you don't have to search around.
- I use boldface text to indicate keywords in bulleted lists or the action parts of numbered steps. I describe many algebraic procedures in a step-by-step format and then use those steps in an example or two.
- Sidebars are shaded boxes that contain text you may find interesting, but this text isn't necessarily critical to your understanding of the chapter or topic.

Foolish Assumptions

Algebra II is essentially a continuation of Algebra I, so I have some assumptions I need to make about anyone who wants (or has) to take algebra one step further.

I assume that a person reading about Algebra II has a grasp of the arithmetic of signed numbers — how to combine positive and negative numbers and come out with the correct sign. Another assumption I make is that your order of operations is in order. Working your way through algebraic equations and expressions requires that you know the rules of order. Imagine yourself at a meeting or in a courtroom. You don't want to be called *out of order!*

I assume that people who complete Algebra I successfully know how to solve equations and do basic graphs. Even though I lightly review these topics in this book, I assume that you have a general knowledge of the necessary procedures. I also assume that you have a handle on the basic terms you run across in Algebra I, such as

- binomial: An expression with two terms
- coefficient: The multiplier or factor of a variable
- constant: A number that doesn't change in value
- expression: Combination of numbers and variables grouped together — not an equation or inequality
- factor (n.): Something multiplying something else
- factor (v.): To change the format of several terms added together into a product

» linear: An expression in which the highest power of any variable term is one

» monomial: An expression with only one term

» polynomial: An expression with several terms

» quadratic: An expression in which the highest power of any variable term is two

» simplify: To change an expression into an equivalent form that you combined, reduced, factored, or otherwise made more useable

» solve: To find the value or values of the variable that makes a statement true

» term: A grouping of constants and variables connected by multiplication, division, or grouping symbols and separated from other constants and variables by addition or subtraction

» trinomial: An expression with three terms

» variable: Something that can have many values (usually represented by a letter to indicate that you have many choices for its value)

If you feel a bit over your head after reading through some chapters, you may want to refer to *Algebra I For Dummies* (Wiley) for a more complete explanation of the basics. My feelings won't be hurt; I wrote that one, too!

Icons Used in This Book

The icons that appear in this book are great for calling attention to what you need to remember or what you need to avoid when doing algebra. Think of the icons as signs along the Algebra II Highway; you pay attention to signs — you don't run them over!

ALGEBRA RULES

This icon provides you with the rules of the road. You can't go anywhere without road signs — and in algebra, you can't get anywhere without following the rules that govern how you deal with operations. In place of "Don't cross the solid yellow line," you see "Reverse the sign when multiplying by a negative." Not following the rules gets you into all sorts of predicaments with the Algebra Police (namely, your instructor).

TIP

This icon is like the sign alerting you to the presence of a sports arena, museum, or historical marker. Use this information to improve your mind, and put the information to work to improve your algebra problem-solving skills.

REMEMBER

This icon lets you know when you've come to a point in the road where you should soak in the information before you proceed. Think of it as stopping to watch an informative sunset. Don't forget that you have another 30 miles to Chicago. Remember to check your answers when working with rational equations.

WARNING

This icon alerts you to common hazards and stumbling blocks that could trip you up — much like “Watch for Falling Rock” or “Railroad Crossing.” Those who have gone before you have found that these items can cause a huge failure in the future if you aren’t careful.

TECHNICAL STUFF

Yes, Algebra II does present some technical items that you may be interested to know. Think of the temperature or odometer gauges on your dashboard. The information they present is helpful, but you can drive without it, so you can simply glance at it and move on if everything is in order.

Beyond the Book

In addition to all the great content provided in this book, you can find even more of it online. Check out `www.dummies.com/cheatsheet/algebraii` for a free Cheat Sheet that provides you with a quick reference to some standard forms, such as special products and equations of conics; some formulas, such as those needed for counting techniques and sequences and series; and, yes, those ever-important laws of logarithms.

You can also find several bonus articles on topics such as just what a *normal* line is (as opposed to *abnormal*?) and how mathematics helped a young man become king at `www.dummies.com/extras/algebraii`.

Where to Go from Here

I’m so pleased that you’re willing, able, and ready to begin an investigation of Algebra II. If you’re so pumped up that you want to tackle the material cover to cover, great! But you don’t have to read the material from page 1 to page 2 and so on. You can go straight to the topic or topics you want or need and refer to earlier material if necessary. You can also jump ahead if so inclined. I include clear cross-references in chapters that point you to the chapter or section where you can find a particular topic — especially if it’s something you need for the material you’re looking at or if it extends or furthers the discussion at hand.

You can use the table of contents at the beginning of the book and the index in the back to navigate your way to the topic that you need to brush up on. Or, if you’re more of a freewheeling type of guy or gal, take your finger, flip open the book, and mark a spot. No matter your motivation or what technique you use to jump into the book, you won’t get lost because you can go in any direction from there.

Enjoy!

1

Homing in on Basic Solutions

IN THIS PART . . .

Get a handle on the basics of simplifying and factoring.

Find out how to get in line with linear equations.

Queue up to quadratic equations.

Take on basic rational and radical equations.

Work through graphing on the coordinate system.

IN THIS CHAPTER

» Abiding by (and using) the rules of algebra

» Adding the multiplication property of zero to your repertoire

» Raising your exponential power

» Looking at special products and factoring

Chapter 1
Going Beyond Beginning Algebra

Algebra is a branch of mathematics that people study before they move on to other areas or branches in mathematics and science. For example, you use the processes and mechanics of algebra in calculus to complete the study of change; you use algebra in probability and statistics to study averages and expectations; and you use algebra in chemistry to work out the balance between chemicals. Algebra all by itself is esthetically pleasing, but it springs to life when used in other applications.

Any study of science or mathematics involves rules and patterns. You approach the subject with the rules and patterns you already know, and you build on those rules with further study. The reward is all the new horizons that open up to you.

Any discussion of algebra presumes that you're using the correct notation and terminology. Algebra I (check out *Algebra I For Dummies* [Wiley]) begins with combining terms correctly, performing operations on signed numbers, and dealing with exponents in an orderly fashion. You also solve the basic types of linear and quadratic equations. Algebra II gets into more types of functions, such as exponential and logarithmic functions, and topics that serve as launching spots for other math courses.

Going into a bit more detail, the basics of algebra include rules for dealing with equations, rules for using and combining terms with exponents, patterns to use when factoring expressions, and a general order for combining all the above. In this chapter, I present these basics so you can further your study of algebra and feel confident in your algebraic ability. Refer to these rules whenever needed as you investigate the many advanced topics in algebra.

Outlining Algebraic Properties

Mathematicians developed the rules and properties you use in algebra so that every student, researcher, curious scholar, and bored geek working on the same problem would get the same answer — no matter the time or place. You don't want the rules changing on you every day (and I don't want to have to write a new book every year!); you want consistency and security, which you get from the strong algebra rules and properties that I present in this section.

Keeping order with the commutative property

ALGEBRA RULES

The commutative property applies to the operations of addition and multiplication. It states that you can change the order of the values in a particular operation without changing the final result:

$a+b=b+a$	Commutative property of addition
$a \cdot b = b \cdot a$	Commutative property of multiplication

If you add 2 and 3, you get 5. If you add 3 and 2, you still get 5. If you multiply 2 times 3, you get 6. If you multiply 3 times 2, you still get 6.

Algebraic expressions usually appear in a particular order, which comes in handy when you have to deal with variables and coefficients (multipliers of variables). The number part comes first, followed by the letters, in alphabetical order. But the beauty of the commutative property is that $2xyz$ is the same as $x2zy$. You have no good reason to write the expression in that second, jumbled order, but it's helpful to know that you can change the order around when you need to.

Maintaining group harmony with the associative property

ALGEBRA RULES

Like the commutative property (see the previous section), the associative property applies only to the operations of addition and multiplication. The associative property states that you can change the grouping of operations without changing the result:

$a+(b+c)=(a+b)+c$	Associative property of addition
$a(b\cdot c)=(a\cdot b)c$	Associative property of multiplication

You can use the associative property of addition or multiplication to your advantage when simplifying expressions. And if you throw in the commutative property when necessary, you have a powerful combination. For instance, when simplifying $(x+14)+(3x+6)$, you can start by dropping the parentheses (thanks to the associative property). You then switch the middle two terms around, using the commutative property of addition. You finish by reassociating the terms with parentheses and combining the like terms:

$$(x+14)+(3x+6)=x+14+3x+6=x+3x+14+6$$
$$=(x+3x)+(14+6)=4x+20$$

The steps in the previous process involve a lot more detail than you really need. You probably did the problem, as I first stated it, in your head. I provide the steps to illustrate how the commutative and associative properties work together; now you can apply them to more complex situations.

Distributing a wealth of values

ALGEBRA RULES

The distributive property states that you can multiply each term in an expression within parentheses by the multiplier outside the parentheses and not change the value of the expression. It takes one operation, multiplication, and spreads it out over terms that you add to and subtract from one another:

$a(b+c)=a\cdot b+a\cdot c$	Distributing multiplication over addition
$a(b-c)=a\cdot b-a\cdot c$	Distributing multiplication over subtraction

For instance, you can use the distributive property on the problem $12\left(\frac{1}{2}+\frac{2}{3}-\frac{3}{4}\right)$ to make your life easier. You distribute the 12 over the fractions by multiplying each fraction by 12 and then combining the results:

$$
\begin{aligned}
&12\left(\frac{1}{2}+\frac{2}{3}-\frac{3}{4}\right)=12\cdot\frac{1}{2}+12\cdot\frac{2}{3}-12\cdot\frac{3}{4}\\
&={}^{6}\cancel{12}\cdot\frac{1}{\cancel{2}_1}+{}^{4}\cancel{12}\cdot\frac{2}{\cancel{3}_1}-{}^{3}\cancel{12}\cdot\frac{3}{\cancel{4}_1}\\
&=6+8-9=5
\end{aligned}
$$

Finding the answer with the distributive property is much easier than changing all the fractions to equivalent fractions with common denominators of 12, combining them, and then multiplying by 12.

TIP

You can use the distributive property to simplify equations — in other words, you can prepare them to be solved. You also do the opposite of the distributive property when you factor expressions; see the section "Implementing Factoring Techniques" later in this chapter.

Checking out an algebraic ID

ALGEBRA RULES

The numbers zero and one have special roles in algebra — as identities. You use identities in algebra when solving equations and simplifying expressions. You need to keep an expression equal to the same value, but you want to change its format, so you use an identity in one way or another:

$a+0=0+a=a$	The additive identity is zero. Adding zero to a number doesn't change that number; it keeps its identity.
$a\cdot 1=1\cdot a=a$	The multiplicative identity is one. Multiplying a number by one doesn't change that number; it keeps its identity.

Applying the additive identity

One situation that calls for the use of the additive identity is when you want to change the format of an expression so you can factor it. For instance, take the expression x^2+6x and add 0 to it. You get x^2+6x+0, which doesn't do much for you (or me, for that matter). But how about replacing that 0 with both 9 and –9? You now have $x^2+6x+9-9$, which you can write as $(x^2+6x+9)-9$ and factor into $(x+3)^2-9$. Why in the world do you want to do this? Go to Chapter 11 and read up on conic sections to see why. By both adding and subtracting 9, you add 0 — the additive identity.

Making multiple identity decisions

You use the multiplicative identity extensively when you work with fractions. Whenever you rewrite fractions with a common denominator, you actually multiply by one. If you want the fraction $\frac{7}{2x}$ to have a denominator of $6x$, for example, you multiply both the numerator and denominator by 3:

$$\frac{7}{2x}\cdot\frac{3}{3}=\frac{21}{6x}$$

Now you're ready to rock and roll with a fraction to your liking.

Singing along in-verses

You face two types of inverses in algebra: additive inverses and multiplicative inverses. The additive inverse matches up with the additive identity and the multiplicative inverse matches up with the multiplicative identity. The additive inverse is connected to zero, and the multiplicative inverse is connected to one.

ALGEBRA RULES

A number and its additive inverse add up to zero. A number and its multiplicative inverse have a product of one. For example, -3 and 3 are additive inverses; the multiplicative inverse of -3 is $-\frac{1}{3}$. Inverses come into play big-time when you're solving equations and want to isolate the variable. You use inverses by adding them to get zero next to the variable or by multiplying them to get one as a multiplier (or coefficient) of the variable.

Ordering Your Operations

When mathematicians switched from words to symbols to describe mathematical processes, their goal was to make dealing with problems as simple as possible; however, at the same time, they wanted everyone to know what was meant by an expression and for everyone to get the same answer to the same problem. Along with the special notation came a special set of rules on how to handle more than one operation in an expression. For instance, if you do the problem $4+3^2-5\cdot 6+\sqrt{23-7}+\frac{14}{2}$, you have to decide when to add, subtract, multiply, divide, take the root, and deal with the exponent.

ALGEBRA RULES

The *order of operations* dictates that you follow this sequence:

1. **Raise to powers or find roots.**
2. **Multiply or divide.**
3. **Add or subtract.**

REMEMBER

If you have to perform more than one operation from the same level, work those operations moving from left to right. If any grouping symbols appear, perform the operation inside the grouping symbols first.

So, to simplify $4+3^2-5\cdot 6+\sqrt{23-7}+\frac{14}{2}$, follow the order of operations:

1. The radical acts like a grouping symbol, so you subtract what's in the radical first: $4+3^2-5\cdot 6+\sqrt{16}+\frac{14}{2}$.
2. Raise the power and find the root: $4+9-5\cdot 6+4+\frac{14}{2}$.
3. Multiply and divide, working from left to right: $4+9-30+4+7$.
4. Add and subtract, moving from left to right: $4+9-30+4+7=-6$.

Zeroing in on the Multiplication Property of Zero

ALGEBRA RULES

You may be thinking that multiplying by zero is no big deal. After all, zero times anything is zero, right? Yes, and that's the big deal. You can use the multiplication property of zero when solving equations. If you can factor an equation — in other words, write it as the product of two or more multipliers — you can apply the multiplication property of zero to solve the equation. The multiplication property of zero states that

> If the product of $a\cdot b\cdot c\cdot d\cdot e\cdot f=0$, at least one of the factors has to represent the number 0.

The only way the product of two or more values can be zero is for at least one of the values to actually be zero. If you multiply (16)(467)(11)(9)(0), the result is 0. It doesn't really matter what the other numbers are — the zero always wins.

The reason this property is so useful when solving equations is that if you want to solve the equation $x^7-16x^5+5x^4-80x^2=0$, for instance, you need the numbers that replace the x's to make the equation a true statement. This particular equation factors into $x^2(x^3+5)(x-4)(x+4)=0$. The product of the four factors shown here is zero. The only way the product can be zero is if one or more of the factors is zero. For instance, if $x=4$, the third factor is zero, and the whole product is zero. Also, if x is zero, the whole product is zero. (Head to Chapters 3 and 8 for more info on factoring and using the multiplication property of zero to solve equations.)

THE BIRTH OF NEGATIVE NUMBERS

In the early days of algebra, negative numbers weren't an accepted entity. Mathematicians had a hard time explaining exactly what the numbers illustrated; it was too tough to come up with concrete examples. One of the first mathematicians to accept negative numbers was Fibonacci, an Italian mathematician. When he was working on a financial problem, he saw that he needed what amounted to a negative number to finish the problem. He described it as a loss and proclaimed, "I have shown this to be insoluble unless it is conceded that the man had a debt."

Expounding on Exponential Rules

Several hundred years ago, mathematicians introduced powers of variables and numbers called exponents. The use of exponents wasn't immediately popular, however. Scholars around the world had to be convinced; eventually, the quick, slick notation of exponents won over, and we benefit from the use today. Instead of writing *xxxxxxxx*, you use the exponent 8 by writing x^8. This form is easier to read and much quicker.

REMEMBER

The expression a^n is an exponential expression with a base of a and an exponent of n. The n tells you how many times you multiply the a times itself.

You use radicals to show roots. When you see $\sqrt{16}$, you know that you're looking for the number that multiplies itself to give you 16. The answer? Four, of course. If you put a small superscript in front of the radical, you denote a cube root, a fourth root, and so on. For instance, $\sqrt[4]{81}=3$, because the number 3 multiplied by itself four times is 81. You can also replace radicals with fractional exponents — terms that make them easier to combine. This use of exponents is very systematic and workable — thanks to the mathematicians who came before us.

Multiplying and dividing exponents

ALGEBRA RULES

When two numbers or variables have the same base, you can multiply or divide those numbers or variables by adding or subtracting their exponents:

- $a^m \cdot a^n = a^{m+n}$: When multiplying numbers with the same base, you add the exponents.
- $\frac{a^m}{a^n} = a^{m-n}$: When dividing numbers with the same base, you subtract the exponents (numerator – denominator). And, in this case, $a \neq 0$.

Also, recall that $a^0 = 1$. Again, $a \neq 0$. To multiply $x^4 \cdot x^5$, for example, you add the exponents: $x^{4+5} = x^9$. When dividing x^8 by x^5, you subtract the exponents: $\frac{x^8}{x^5} = x^{8-5} = x^3$. You must be sure that the bases of the expressions are the same. You can multiply 3^2 and 3^4, but you can't use the rule of exponents when multiplying 3^2 and 4^3.

Getting to the roots of exponents

ALGEBRA RULES

Radical expressions — such as square roots, cube roots, fourth roots, and so on — appear with a radical to show the root. Another way you can write these values is by using fractional exponents. You'll have an easier time combining variables with the same base if they have fractional exponents in place of radical forms:

- $\sqrt[n]{x} = x^{1/n}$: The root goes in the denominator of the fractional exponent.
- $\sqrt[n]{x^m} = x^{m/n}$: The root goes in the denominator of the fractional exponent, and the power goes in the numerator.

So, you can say $\sqrt{x} = x^{1/2}$, $\sqrt[3]{x} = x^{1/3}$, $\sqrt[4]{x} = x^{1/4}$, and so on, along with $\sqrt[5]{x^3} = x^{3/5}$.

To simplify a radical expression such as $\frac{\sqrt[4]{x}\sqrt[6]{x^{11}}}{\sqrt[2]{x^3}}$, you change the radicals to exponents and apply the rules for multiplication and division of values with the same base (see the previous section):

$$\begin{aligned}\frac{\sqrt[4]{x}\sqrt[6]{x^{11}}}{\sqrt[2]{x^3}} &= \frac{x^{1/4} \cdot x^{11/6}}{x^{3/2}} = \frac{x^{1/4+11/6}}{x^{3/2}} = \frac{x^{3/12+22/12}}{x^{18/12}} \\ &= \frac{x^{25/12}}{x^{18/12}} = x^{25/12-18/12} = x^{7/12} = \sqrt[12]{x^7}\end{aligned}$$

Raising or lowering the roof with exponents

ALGEBRA RULES

You can raise numbers or variables with exponents to higher powers or reduce them to lower powers by taking roots. When raising a power to a power, you multiply the exponents. When taking the root of a power, you divide the exponents:

- $(a^m)^n = a^{m \cdot n}$: Raise a power to a power by multiplying the exponents.
- $\sqrt[m]{a^n} = \left(a^n\right)^{1/m} = a^{n/m}$: Reduce the power when taking a root by dividing the exponents.